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DECOUPLING OF TWO-TIME-SCALE LINEAR SYSTEMS.(U)
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DECOUPLING OF TWO-TIME-SCALE LINEAR SYSTEMS

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ABSTRACT

Recent studies of two-time-scale linear systems have used a simple transformation of variables to block-diagonalize the system so that fast and slow modes are decoupled. A matrix used in this transformation satisfies an algebraic Riccati equation. In this study it is shown that this transformation is unique and its eigenstructure is described. Also, a new method of computing the decoupling transformation is presented and demonstrated using the 16th-order linear model of a turbofan engine.

INTRODUCTION

Consider the system of linear first-order differential equations

$$\dot{x} = Ax \tag{1}$$

where x is an n-dimensional vector and A is a real, constant matrix. As proposed by Chow and Kokotovic (1976), the system (1) will be called <u>two-time-scale</u> if the eigenvalues of A, represented as $\lambda(A)$, can be separated by absolute value into nonempty sets S and F so that

$$|s_i| \ll |f_j| \quad \text{for all} \quad s_i \in S \quad \text{and} \quad f_j \in F. \tag{2}$$

Let n_1 be the number of eigenvalues in S, and order them so that

$$|s_i| \leq |s_{i+1}|, \quad i = 1, 2, \ldots n_1 - 1$$

$$|f_j| \leq |f_{j+1}|, \quad j = 1, 2, \ldots n_2 - 1,$$

where $n_2 = n - n_1$.

For any two-time-scale system the ratio

$$\mu = \frac{|s_{n_1}|}{|f_1|} \ll 1 \tag{3}$$

will define the system <u>small parameter</u> needed to measure the system's time-scale separation. Such systems are also described as stiff, ill-conditioned or singularly perturbed.

Given the system with n_1 slow modes and n_2 fast modes, partition the state vector x into subvectors x_1 and x_2 of dimension n_1 and n_2, and partition the A matrix accordingly so (1) can be written

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{4}$$

The following two step transformation has been used by Chang (1972) and Kokotovic (1975) to first reduce (4) to block-triangular form and then to block-diagonal form. If the $n_2 \times n_1$ matrix L satisfies the nonsymmetric algebraic Riccati equation

$$LA_{11} + A_{21} - LA_{12}L - A_{22}L = 0, \tag{5}$$

then the first transformation of variables

$$\begin{bmatrix} x_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ L & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{6}$$

reduces (4) to

$$\begin{bmatrix} \dot{x}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} B_1 & A_{12} \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_2 \end{bmatrix} \tag{7}$$

where $B_1 = A_{11} - A_{12}L$ and $B_2 = A_{22} + LA_{12}$. Next, if the $n_1 \times n_2$ matrix K satisfies the Lyapunov equation

$$KB_2 - B_1K + A_{12} = 0 \tag{8}$$

then the second transformation

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I & K \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ y_2 \end{bmatrix} \tag{9}$$

reduces (7) to

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} y_2 \\ y_2 \end{bmatrix}. \tag{10}$$

Since (7) is obtained from (4) by a nonsingular linear transformation and is block-triangular, the eigenvalues of A are divided between B_1 and B_2. That is, $\lambda(B_1) \cup \lambda(B_2) = \lambda(A)$. If we can find a matrix L satisfying (5) such that

$$\lambda(B_1) = S, \quad \lambda(B_2) = F \tag{11}$$

then we have achieved a decoupling of the system slow and fast modes. This decoupled form can be very useful when working with large linear systems and aid in numerical simulation, reduced-order modeling, and control system design.

One attractive feature of the transformation defined by (6) and (9)

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I+KL & K \\ L & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{12}$$

is that the inverse transformation is simply given by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} I & -K \\ -L & I+LK \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{13}$$

which requires no matrix inversion. Also, y_1 represents the exact slow component of the x_1 variables, so results obtained in the decoupled form (10) are easily interpreted.

THE EIGENSTRUCTURE OF L AND K

In order to describe the L and K matrices, the A matrix will be represented in Jordan form

$$A = M\,J\,Q = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \tag{14}$$

where $Q = M^{-1}$. The M, J and Q matrices are partitioned compatibly with the partitioned system (4).

If the A matrix is nondefective, i.e., has a full set of linearly independent eigenvectors, then J is diagonal with eigenvalues appearing along the diagonal. The M matrix is the modal matrix whose columns are the corresponding (right) eigenvectors of A. If A is defective then J contains one or more Jordan blocks and some of the columns of M are generalized eigenvectors. Q is the matrix of reciprocal basis vectors whose rows are the left eigenvectors and left generalized eigenvectors of A. The existence and uniqueness of M and J are discussed in many linear algebra texts such as Stewart (1973).

For the case where A has multiple eigenvalues it can be difficult to get a unique representation of M. However, if λ is an eigenvalue with multiplicity m, there is a unique m-dimensional subspace spanned by the eigenvectors and generalized eigenvectors corresponding to λ.

Narasimhamurthi and Wu (1977) and Veljko (1977) have described the eigenstructure of the solution to the algebraic Riccati equation (5). Those results are given below as Theorem 1 with a simplified proof. Potter (1966), Martensson (1971) and Kucera (1972) presented similar results for the symmetric algebraic Riccati equation which arises from the linear regulator problem.

In Theorem 1 we need not assume that A is two-time-scale, but only that partition (14) is possible. That is, that the Jordan matrix J can be partitioned into submatrices J_1, J_2 without splitting a Jordan block. Also, the L matrix may be complex.

Theorem 1. Matrix L satisfies the algebraic Riccati equation (5) if and only if L satisfies the linear equation

$$LM_{11} = -M_{21} \tag{15}$$

for $A = M J Q$ partitioned as (14).

Proof: Assume first that (5) is satisfied. Rewrite (5) as $L(A_{11} - A_{12}L) = -A_{21} + A_{22}L$ and let $(A_{11} - A_{12}L)$ have Jordan form XGX^{-1}. Setting $Y = -LX$, it follows that

$$A_{11} - A_{12}L = XGX^{-1}$$

$$A_{21} - A_{22}L = -LXGX^{-1}.$$

Post-multiply by X to obtain

$$A_{11}X + A_{12}Y = XG$$

$$A_{21}X + A_{22}Y = YG$$

or

$$A \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} G.$$

Thus the diagonal elements of the Jordan form G are n_1 of the n eigenvalues of A, and

$$\begin{bmatrix} X \\ Y \end{bmatrix}$$

are n_1 corresponding eigenvectors of A. Also $LX = -Y$, completing the first half of the proof.

Suppose $LM_{11} = -M_{21}$. We need to show that the lower left block of the product

$$\begin{bmatrix} L & 0 \\ L & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -L & I \end{bmatrix} = \begin{bmatrix} A_{11}-A_{12}L & A_{12} \\ LA_{11}+A_{21}-LA_{12}L-A_{22}L & A_{22}+LA_{12} \end{bmatrix}$$

is zero.

Since M is full rank, the matrices

$$\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} = \begin{bmatrix} I \\ -L \end{bmatrix} M_{11} \quad \text{and} \quad M_{11}$$

are both full rank and L can be represented as $-M_{21}M_{11}^{-1}$. Also, since $Q\,M = I$, it follows that

$$Q_{21}M_{11} + Q_{22}M_{21} = 0$$

so the L matrix can also be represented as

$$Q_{22}L = Q_{21} \quad \text{or} \quad L = Q_{22}^{-1}Q_{21}. \tag{16}$$

Write A in the Jordan form and complete the product so that

$$\begin{bmatrix} I & 0 \\ -M_{21}M_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -Q_{22}^{-1}Q_{21} & I \end{bmatrix} =$$

$$\begin{bmatrix} M_{11}J_1M_{11}^{-1} & M_{11}J_1Q_{12}+M_{12}J_2Q_{22} \\ 0 & Q_{22}^{-1}J_2Q_{22} \end{bmatrix} = \begin{bmatrix} B_1 & A_{12} \\ 0 & B_2 \end{bmatrix},$$

completing the proof.

In the final expression we use the identities

$$M_{11}^{-1} = Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}$$

and

$$Q_{22}^{-1} = M_{22} - M_{21}M_{11}^{-1}M_{12}.$$

As this theorem illustrates, if L_{-1} satisfies (15), then the eigenvalues and eigenvectors of B_1 and B_2 are M_{11}, J_1 and Q_{22}, J_2 respectively. The case of generalized eigenvectors causes no difficulty in the proof since any Jordan block appearing in G also appears in J_1, and vice versa.

The conditions which insure that L is real also follow quite readily from this theorem. That is, if A has complex eigenvalue λ appearing in J_1, then λ and $\bar{\lambda}$ † must only appear in J_1. Under this restriction there will always exist a nonsingular complex matrix C such that

$$\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} C = \begin{bmatrix} M'_{11} \\ M'_{21} \end{bmatrix}$$

is real. If L satisfies $LM_{11} = -M_{21}$ then L also satisfies $LM'_{11} = M'_{21}$ so that L is real.

The general form of the block-diagonalization transformation (12) can be determined from Theorem 1. Comparing the Jordan forms $A = M\,J\,Q$ and

$$\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} = \begin{bmatrix} M_{11} & 0 \\ 0 & Q_{22}^{-1} \end{bmatrix} \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} M_{11}^{-1} & 0 \\ 0 & Q_{22} \end{bmatrix}$$

it is apparent that

$$\begin{bmatrix} I+KL & K \\ L & I \end{bmatrix} = \begin{bmatrix} M_{11}Q_{11} & M_{11}Q_{12} \\ Q_{22}^{-1}Q_{21} & I \end{bmatrix}.$$

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† complex conjugate

UNIQUENESS OF THE L AND K MATRICES

As shown by Narasimhamurthi and Wu (1977), only under very restrictive conditions will the algebraic Riccati equation have a unique solution. Namely, only when the quadratic term is zero and the equation becomes linear. However, there is a unique solution which decouples a two-time-scale system with slow and fast modes separated. This is proven in the following theorem.

Theorem 2. For a given two-time-scale system A there is a unique decoupling matrix L satisfying both (5) and (11).

Proof: Assume that L and L' both satisfy (5) and (11). Applying Theorem 1 there exist full rank matrices

$$\begin{bmatrix} X \\ Y \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} X' \\ Y' \end{bmatrix} \tag{17}$$

such that $LX = -Y$ and $L'X' = -Y'$. Moreover, the columns are (generalized) eigenvectors of A corresponding to eigenvalues S. Since there is a unique n_1-dimensional subspace spanned by these (generalized) eigenvectors, there exists matrix C such that

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X' \\ Y' \end{bmatrix} C.$$

Post-multiply $L'X' = -Y'$ by C to obtain $L'X = -Y$. As shown in the proof of Theorem 1, X^{-1} exists. Thus $L' = L = -YX^{-1}$, completing the proof.

Although the decoupling matrix L is unique for a given two-time-scale system, there are generally many real solutions to the algebraic Riccati equation (5). Thus any effort to obtain L should avoid these other solutions. In a special case the number of distinct solutions can be enumerated by

Corollary 1. If A has distinct real eigenvalues, then there are $\binom{n}{n_1}$ distinct real solutions of (5) corresponding to the number of distinct ways of partitioning the n eigenvalues of A into sets of n_1 and n_2 members.

Proof: Since M is real, all solutions of (5) are real. We will assume that $\lambda(A_{11}-A_{12}L) \neq \lambda(A_{11}-A_{12}L')$ and prove that this implies $L \neq L'$. Let $LX = -Y$ and $L'X' = -Y'$. The linear independence of the eigenvectors corresponding to the $n_1 + 1$ or more eigenvalues $\lambda(A_{11}-A_{12}L) \cup \lambda(A_{11}-A_{12}L')$ implies that the augmented matrix

$$\begin{bmatrix} X & X' \\ Y & Y' \end{bmatrix} \tag{18}$$

has rank at least $n_1 + 1$. Then $L = -YX^{-1}$ and $L' = -Y'(X')^{-1}$. If we assume that $L = L'$ so that $YX^{-1} = Y'(X')^{-1}$ and $Y = Y'(x')^{-1}X$, then

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y'(X')^{-1}X \end{bmatrix} = \begin{bmatrix} X' \\ Y' \end{bmatrix} (X')^{-1}X$$

which implies that the n_1 columns of $\begin{bmatrix} X \\ Y \end{bmatrix}$ are linear combinations of the columns of $\begin{bmatrix} X' \\ Y' \end{bmatrix}$ and the rank of (18) equals n_1, a contradiction. Thus $L \neq L'$ and the proof is complete.

The uniqueness of the K matrix is insured by the two-time-scale property. As shown by Gantmacher (1959), the linear equation (8) has a unique solution provided B_1 and B_2 have no common eigenvalues.

THE CHOICE OF SLOW AND FAST VARIABLES

An important question which arises in the decoupling and reducer-order modeling of large linear systems is how should state variables x be reordered so that states which contain predominately slow mode are placed in x_1. If we reorder the state variables then the rows of the modal matrix M are reordered in the same manner. From our previous results it is apparent that $\|L\|$ is approximately proportional to $\|M_{21}\|$ and inversely proportional to $\|M_{11}\|$. So if we can choose an ordering which minimizes $\|M_{11}\|\|M_{21}\|$, this will insure that the

slow modes appear primarily in variables x_1, and $\|L\|$ will be small. In a similar fashion, the K matrix can be written as

$$K = -M_{12}Q_{22}$$

so that choosing an ordering of states which minimizes $\|M_{12}\|$ will place primarily fast variables in x_2 and yield a small $\|K\|$.

This process is clarified by introducing slow mode and fast mode coupling ratios ρ_s and ρ_f. Assume that the columns of M are normalized so that each is of length one, and define

$$\rho_s = \frac{|M_{21}|}{|M_{11}|}, \qquad \rho_f = \frac{|M_{12}|}{|M_{22}|}.$$

Here $|\cdot|$ refers to the Euclidean norm and, where necessary, take the modulus of complex entries. Either ρ_s or ρ_f can be minimized by searching over all orderings of the row indices $\{i\}$ of the modal matrix $M = [m_{ij}]$. If it should happen that all rows of

$$\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}$$

have equal norm, then the minimum value of ρ_s will be $(n_2/n_1)^{1/2}$. On the other hand, if x can be ordered so that $M_{21} = 0$, then $\rho_s = 0$ and the slow modes are naturally decoupled from the x_2 states. For this ordering of x, the A matrix will be block-triangular. Likewise, if $M_{12} = 0$, then ρ_f is zero and the resulting A will be block-triangular.

Depending upon the structure of M, the ordering which minimizes ρ_s may or may not also minimize ρ_f. Depending upon whether we are primarily interested in studying the slow behavior or fast behavior of the system, we could choose the ordering of variables which minimizes either ρ_s or ρ_f.

If both ρ_s and ρ_f are small, we could describe the system as being "weakly coupled." This definition of weakly coupled systems is similar to one proposed by Milne (1965). However, his definition also required that μ be small. Even though a system is strongly two-time-scale, it need not therefore be weakly coupled.

Provided one is able to compute eigenvectors, the following procedure is suggested for reindexing x, A and M before obtaining the decoupling transformation.

Algorithm 1.

1. Normalize the columns of

$$\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} \tag{19}$$

so that $\sum_{i=1}^{n} |m_{ij}|^2 = 1$, $j = 1, \ldots n_1$.

2. If (19) contains one or more complex eigenvectors v_i, replace v_i with $\mathrm{Re}(v_i)$ and replace v_i with $\mathrm{Im}(v_i)$.

3. Evaluate the norm of each of the n rows of (19) and let

$$r_i = \left(\sum_{j=1}^{n_1} m_{ij}^2\right)^{1/2}, \quad i = 1, \ldots n.$$

4. Order the set $\{r_i\}$ so that $r_{\alpha_1} \geq r_{\alpha_2} \geq \ldots \geq r_{\alpha_n}$ and premultiply x, A and M by the permutation matrix

$$\begin{bmatrix} e_{\alpha_1} & e_{\alpha_2} & \cdots & e_{\alpha_n} \end{bmatrix} \tag{20}$$

where e_i is the i-th column of the $n \times n$ identity matrix.

5. Post-multiply A by the inverse permutation matrix

$$\left[e_{\alpha_1} \quad e_{\alpha_2} \quad \cdots \quad e_{\alpha_n} \right]^T$$

where $[\cdot]^T$ indicates transpose.

This general method for systematically picking slow variables should be very useful in applications as discussed by Teneketzis and Sandell (1976). In general not all eigenvectors are required, but only those corresponding to the slow modes. However, one question which needs further exploration is how scaling of the state variables influences this method.

COMPUTING THE L AND K MATRICES

Kokotovic (1975) proposed a method of computing the L matrix starting with initial approximation

$$L_o = A_{22}^{-1}A_{21} \tag{21}$$

and using the iteration

$$L_{i+1} = A_{22}^{-1}(L_i(A_{11} - A_{12}L_i) + A_{21}). \tag{22}$$

He proved that if a certain criterion involving matrix norms is satisfied then this iteration will converge to L. As he noted, the criterion is quite restrictive and is not satisfied by many typical two-time-scale systems.

If efficient eigenanalysis programs are available, then a computational procedure for finding accurate numerical L and K matrices based on the results of Section 2 is quite straightforward. Examples of such programs are EISPACK from the Argonne National Laboratory, described by Smith et al. (1976), or EIGRF from the International Mathematics and Statistics Library (IMSL), Houston, Texas. These eigenanalysis programs are well documented and easy to implement on mainframe computers. Typical execution times for finding all eigenvalues and eigenvectors of matrices of order 40 and 80 are .66 and 4.6 seconds using an IBM 370/195 computer, cf. Smith et al. (1976). Both programs are based upon the QR algorithm, cf. Wilkenson (1965).

Since the L matrix can be expressed as either $-M_{21}M_{11}^{-1}$ or $Q_{22}^{-1}Q_{21}$, not all eigenvectors of A need be evaluated to solve for L. Two procedures could be used, one for the case $n_1 \leq n_2$ and another for $n_1 > n_2$. For $n_1 \leq n_2$ obtain n_1 (right) eigenvectors corresponding to the slow eigenvalues S and solve the set of linear equations (15). Such equations can be efficiently solved by some Gaussian elimination scheme such as LU decomposition, e.g., Stewart (1973). For $n_1 > n_2$, find the n_2 left eigenvectors $(Q_{21}Q_{22})$ corresponding to the fast eigenvalues F and solve the linear system (16). In the event that complex eigenvectors are encountered, apply step 2 of Algorithm 1 to eliminate complex entries in (15) or (16).

The accuracy of the matrix L_o obtained from (15) or (16) can be evaluated by substituting it back into the Riccati equation (5) and evaluating the residual error matrix

$$R_o = L_oA_{11} + A_{21} - L_oA_{12}L_o - A_{22}L_o.$$

If R_o is judged to be not sufficiently small, i.e., $\|R_o\| > \varepsilon$ for some small number ε, then the accuracy of L_o can be improved by an iterative procedure. If L_o is a reasonable approximation to L then the matrix $(A_{22} + L_oA_{12})$ has large eigenvalues, is nonsingular, and suggests the linear iteration

$$L_{i+1} = (A_{22} + L_iA_{12})^{-1}(L_iA_{11} + A_{21}).$$

This iteration can be rewritten as

Algorithm 2.

1. Obtain initial approximation L_o from either (15) or (16) and set $i = 0$.
2. Evaluate $R_i = L_iA_{11} + A_{21} - L_iA_{12}L_i - A_{22}L_i$ and stop if $\|R_i\| \leq \varepsilon$.
3. Solve $(A_{22} + L_iA_{12})D_i = R_i$ for D_i and let $L_{i+1} = L_i + D_i$.
4. Let $i = i + 1$ and go to 2.

It can be shown that for L_o close to L, the proposed iteration converges with approximate rate μ. That is,

$$R_{i+1} = (L_i + D_i)A_{11} + A_{21} - (L_i + D_i)A_{12}(L_i + D_i) - A_{22}(L_i + D_i)$$

$$R_{i+1} = R_i + D_i(A_{11} - A_{12}L) - (A_{22} + L_iA_{12})D_i - D_iA_{12}D_i$$

and taking the correction matrix D_i as in step 3 above

$$R_{i+1} = (A_{22} + L_iA_{12})^{-1}R_i(A_{11} - A_{12}L_i - A_{12}D_i)$$

$$= (A_{22} + L_iA_{12})^{-1}R_i(A_{11} - A_{12}L_{i+1}).$$

The largest eigenvalues of $(A_{22} + LA_{12})^{-1}$ and $(A_{11} - A_{12}L)$ are $1/f_1$ and s_{n_1}, so that if L_o is close to L, then

$$\frac{\|R_{i+1}\|}{\|R_i\|} \sim \frac{|s_{n_1}|}{|f_1|} = \mu << 1.$$

The convergence of the iteration defined by Algorithm 2 could be insured by including a line search in step 3. That is, if $\|R_{i+1}\| > \|R_i\|$, then let $L_{i+1} = L_i + hD_i$ and search on h until $\|R_{i+1}\|$ is minimized. However, this safeguard procedure was not needed in the example which follows.

This procedure for obtaining numerical solutions to the nonsymmetric Riccati equation has some similarities to one proposed by Farrar and Di Pietro (1977) for solving the symmetric Riccati equation arising in the linear quadratic regulator problem in that both procedures are initialized using eigenvectors. However, their iterative correction involves a bilinear equation whereas Algorithm 2 involves a linear iterative correction.

Algorithm 3 listed below can be used to compute the K matrix. Based on the previous argument that the eigenvalues of B_2 are all larger than those of B_1, it can be shown that the rate of convergence will approximately equal the system small parameter.

Algorithm 3.

1. Set $K_o = 0$, $R_o = -A_{12}$, and $i = 0$.

2. Solve $D_iB_2 = R_i$ for D_i and let $K_{i+1} = K_i + D_i$.

3. Evaluate $R_{i+1} = K_{i+1}B_2 - B_1K_{i+1} + A_{12}$ and stop if $\|R_{i+1}\| \leq \epsilon$.

4. Set $i = i + 1$ and go to step 2.

An advantage of this approach is that B_2 is unchanged in the iteration so that it can be decomposed once into LU form and then used repeatedly in step 2 to find D_i.

To summarize, the proposed procedure for computing the decoupling transformation for a two-time-scale system involves the following steps.

1. Obtain approximate eigenvalues $\lambda(A)$ and choose the number of slow modes n_1. Note that different choices of n_1 are possible to suit the needs of the study.

2. If $n_1 \leq n_2$ obtain approximate eigenvectors $\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}$. Otherwise, obtain left eigenvectors $[Q_{21} \; Q_{22}]$.

3. (Optional) Apply Algorithm 1 to place slow variables in x_1.

4. Eliminate imaginary parts as needed, and solve either (15) or (16) for L_o.

5. (Optional) Improve the accuracy of L_o by Algorithm 2.

6. Find K by Algorithm 3.

This procedure has been successfully applied to five models of physical systems with orders ranging from 4 to 32. In all cases Algorithms 2 and 3 converged with approximate rate μ and it was verified that the eigenvalue decoupling conditions (11) were satisfied. However, for the sake of brevity, only one of these examples will be presented here.

TURBOFAN EXAMPLE

This example is a 16th order model of a turbofan engine which was the theme problem for a recent conference on control of linear multivariable systems, i.e., Sain (1977). This model is the linearization of a detailed nonlinear simulation of the turbofan at the sea level static intermediate thrust operating point. Two of the state variables are shaft speeds, three are pressures and eleven are temperatures. The eigenvalues for the linear system are -0.65, -1.90, -2.62, $-6.72 \pm j1.31$, $-17.8 \pm j4.8$, -18.6, $-21.3 \pm j0.8$, -38.7, -47.1, -50.7, -59.2, -175.7, and -577.0. For $n_1 = 15$, 5 and 3, the resulting small parameters are 0.304, 0.371, and 0.383. Since these values are not particularly small relative to one, we might call this system weakly two-time-scale.

The computations were performed at the University of Arizona Computing Center on both the DEC 10 and CYBER 175 computers to compare the effects of computer word length and compare execution times for time sharing against batch processing.

The performance of Algorithms 2 and 3 for the two cases $n_1 = 3$ and $n_1 = 5$ is illustrated in Figures 1 through 4. The state variables were not reordered for this computation, i.e., Algorithm 1 was not employed. As shown, in all cases the convergence rate approximately equals the small parameter. The corresponding computation times are listed in Table 1. The six orders-of-magnitude difference between the minimum residual errors obtained with the DEC 10 and CYBER computers can be directly attributed to their 36 and 64 bit word lengths. Both EISPACK and IMSL were tested in the computation of L_0. Their performance in terms of eigenvalue accuracy was found to be very similar, with ISML yielding somewhat shorter execution times. The execution times in Table 1 were obtained using EISPACK.

The effects of reordering the state variables before partitioning x into $(x_1,x_2)^T$ are illustrated in Figure 5 for the case $n_1 = 5$. As shown, the choice of ordering does not in this case affect the rate of convergence of Algorithm 2, but strongly influences ρ_s and $\|L\|$.

The computation of the L matrix using alternate initial values L_0 is described in Figure 6. It's interesting to note that Algorithm 2 succeeds in converging to the correct solution when initialized with $L_0 = A_{22}^{-1}A_{21}$, i.e., equation (21), or with $L_0 = 0$. This demonstrates that in some cases the decoupling transformation can be obtained without the use of eigenvectors. However, the L_0 matrix obtained using eigenvectors provides a much more reliable initial value for Algorithm 2.

CONCLUSIONS

This work considers the properties of a transformation used by Chang (1972) and Kokotovic (1975) to transform linear systems into block-diagonal form with slow and fast modes decoupled. The eigenstructure of this transformation is described, and it is shown that for any given two-time-scale linear system the decoupling transformation is unique.

General conditions for the existence of solutions of the nonsymmetric algebraic Riccati equation are established which are more general than existence conditions previously stated by Narasimhamurthi and Wu (1977). It is also shown that when the corresponding A matrix has distinct real eigenvalues, the distinct real solutions to the Riccati equation can be specifically enumerated.

A new method of computing the decoupling transformation is proposed. With this method the number of slow modes can be chosen to meet the needs of the study provided multiple or complex eigenvalues are not separated. The method includes algorithms for selecting slow system variables, and computing accurate L and K matrices. Although system eigenvectors are used in this method, not all eigenvectors need be found, and highly accurate eigenvectors are not required. A 16th order linear model of a turbofan is decoupled by this method to demonstrate computation times, convergence rates, and algorithm stability.

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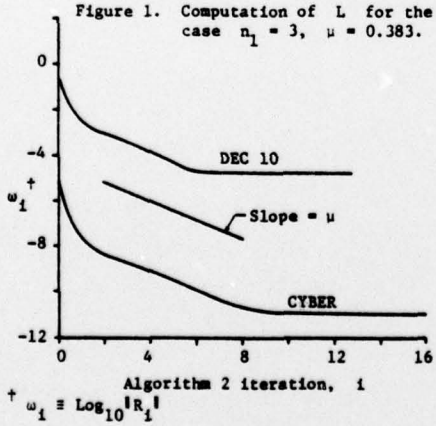
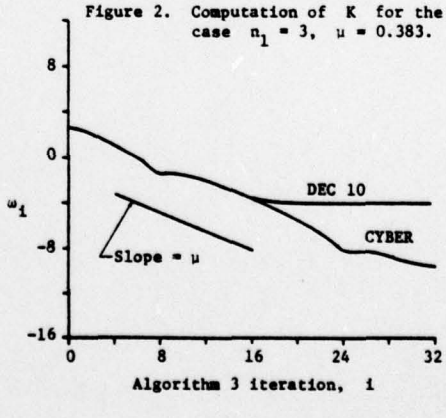
Figure 1. Computation of L for the case $n_1 = 3$, $\mu = 0.383$.



† $\omega_i \equiv \text{Log}_{10}\|R_i\|$

Figure 2. Computation of K for the case $n_1 = 3$, $\mu = 0.383$.



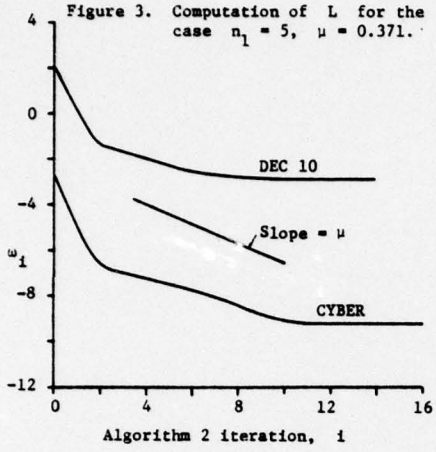


Figure 3. Computation of L for the case $n_1 = 5$, $\mu = 0.371$.

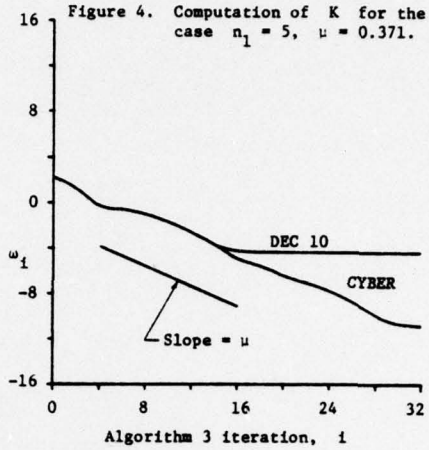


Figure 4. Computation of K for the case $n_1 = 5$, $\mu = 0.371$.

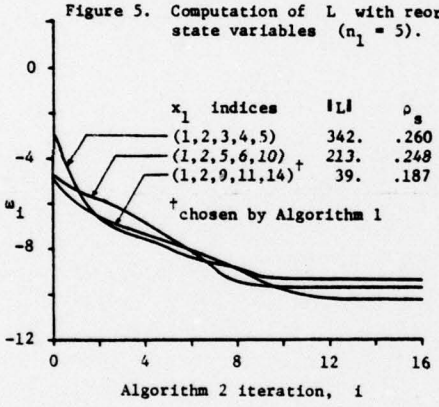


Figure 5. Computation of L with reordered state variables ($n_1 = 5$).

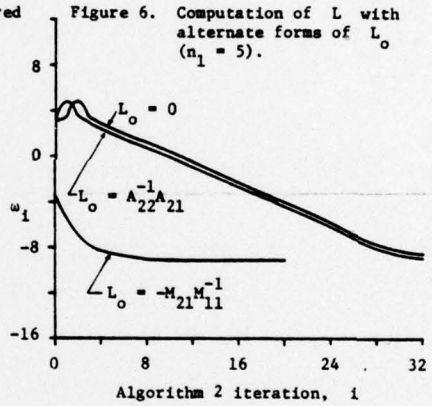


Figure 6. Computation of L with alternate forms of L_o ($n_1 = 5$).

Table 1. Typical Execution Times
(CPU seconds)

	Case $n_1 = 5$		Case $n_1 = 3$	
	CYBER	DEC 10	CYBER	DEC 10
Time to compute L_o	.109	.615	.096	.599
Time per L_i iteration (Algorithm 2)	.023	.187	.021	.171
Time per K_i iteration (Algorithm 3)	.015	.126	.012	.091

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Recent studies of two-time-scale linear systems have used a simple transformation of variables to block-diagonalize the system so that fast and slow modes are decoupled. A matrix used in this transformation satisfies an algebraic Riccati equation. In this study it is shown that this transformation is unique and its eigenstructure is described. Also, a new method of computing the decoupling transformation is presented and demonstrated using the 16th-order linear model of a turbofan engine.

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